Build a Better Tesseract and the World Will Beat a (Hyper) Path to Your Door

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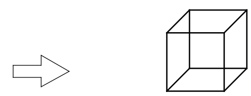
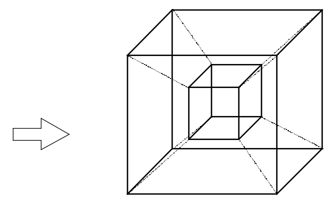
The difficulty of some games is the need to juggle multiple dimensions of information in one’s head. In this paper, we construct a hypercube diagram to clarify 4D relationships and then discuss its application to several well know games.

The Perfect Hypercube

We wish to render a 4D object in 2D. Typically, an extra dimension in a drawing can be abstracted "into the paper" by using perspective. To make a 4D drawing, we must have two dimensions of perspective without allowing them to become confused.

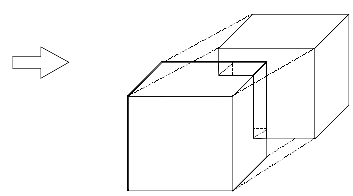


The typical drawing of a hypercube (hcube) is a cube inside another cube with corresponding corners connected. In this case the additional perspective dimension is represented by the distinction between inside and outside.

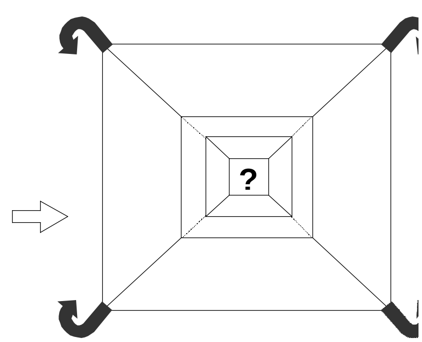


This rendering is a rather odd perspective however. It would be equivalent to drawing a cube end-on, i.e., a square inside a square.

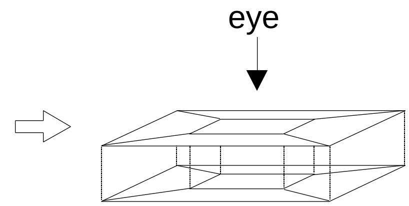
A more representative view of a hcube would be analogous to the first illustration. I.e. overlap the two cubes instead of nesting them. But because the cubes overlap, some of their faces interpenetrate. As in the case of the first cube, this an illusion due to perspective, but being confined to 3D as we are, the results are confusing to our eyes.

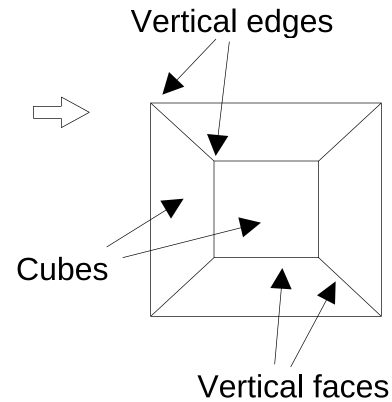


The cube-in-cube rendering avoids this collision trouble, but still becomes fouled when projecting from 3 dimensions to two. Before sitting down with markers and string to try and untangle it, topologists will happily come by and prove to us that it can't be done. The best drawing that can be made is something like an end-on cube inside another end-on cube. One set of corners connects all right, but the four outside corners need to go all that way inside, and these queer the deal.



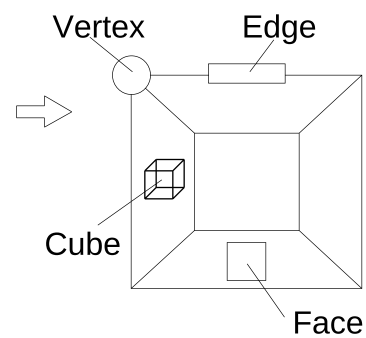
So we try a different tack. We note that the previous illustration was simply two flattened cubes whose corners needed connection. Suppose we were to superimpose these two cubes and look down from above? We would see a doubled drawing. Each vertex would now stand for two corners of the hcube, one over the other. Every edge and face would likewise be doubled. The connecting edges between the two levels would be seen end on, and so appear as points, one at each vertex of the illustration. Likewise, the faces these edges bound would also be vertical and so appear as lines, one per edge of the picture. Finally, the five faces of the illustration sit above five of the hcube's composing cubes. Recalling that a hcube is supposed to contain *eight* cubes in all, we make a note to go looking for the missing three shortly.



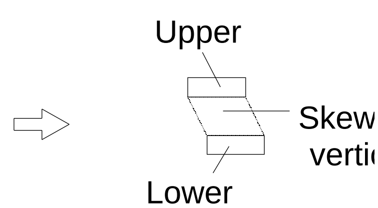


Our intention is to make a diagram that depicts all features of an hcube and their interrelationships. This requires each element of our superimposed model to present a triple of information: the pair of elements from the upper and lower levels, and the vertical element they define between them.

We choose to place a proxy in the center of each element to represent the object it sits on. This avoids the difficulty of indicating exactly what thing is being labeled in a given location. Our flattened cube is representing its third dimension as "outside vs. inside," enabling us to use normal perspective to offset our "proxy triples." We must still make sure however, that they remain closely packed so they will appear as a unit.

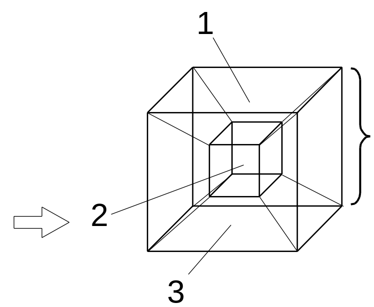


The illustration at the end of the paper represents our best attempt to do these things. It presents all elements composing a hypercube in a semblance of coherent fashion. Elements on the lower level are offset down and right from upper ones, with the vertical element sandwiched between them. The vertical elements are skewed by the offset to provide a sense of depth.



All proxies for a type of feature are assigned the same color. However, those on the upper level are given a lighter shade. Proper occlusion is provided whenever possible and reference lines have been drawn between cubes and their constituent faces.

We now return to the question of the three missing cubes. It turns out that the central square contains not one cube, but *four*. Recall that the upper and lower levels of our diagram are themselves flattened cubes. They did not cease being cubes when we pushed them flat. Therefore, in addition to the central cube that they define between them, they each contain a very thin cube inside.



Examination of the traditional hcube drawing shows us the same thing. We see that there a total of three cubes stacked through the center, albeit the highest and lowest ones distorted. This illustration also shows us where the last one is hiding. The entire outside of the figure is a cube! Recall that this illustration was originally constructed by drawing a cube inside a cube. We already accounted for the one inside, but the outside cube remained.

It's hard to draw the central stack of features accurately. We have represented it as three cubes sandwiched between four faces. We have added some scaling to help the top and bottom faces stand away from the rest. There is no reasonable location for the final "external" cube. It rightly belongs outside the entire drawing or co-located with the central cube. As a compromise, we have placed it in the central area, off to one side. We use dotted lines to point to its top and bottom faces, and arrows to point to the rest, inconveniently located on the outside of the illustration.

4D in the Game of Quarto

Quarto consists of a 4x4 board and 16 unique pieces. The pieces represent all combinations of four feature-pairs: tall/short, black/white, round/square, and divot/flat. Obviously with four orthogonal features, the pieces map to the 16 corners of our hcube.

Players alternate selecting a piece for their opponent to play on an empty square of the board. A win consists of a row of four pieces all sharing any one feature. Therefore there are 8 kinds of winning rows each of which can be played on any of the 10 rows/columns/diagonals of the board.

The game has two difficulties: managing the set of unplayed pieces in one's head, and mapping how this relates to threats on the board. Our hcube can help with these.

If we are playing pieces from hcube corners onto the board, this is equivalent to selecting board squares and playing them onto the corners of the hcube. The result will in an isomorphic game whose board has 4D complexity, leaving us to contemplate only a 2D set of information. This leads to a more general axiom:

Put the complexity on the board, not in your head

We call our new game Duo. We assign a color to each of the ten rows/columns/diagonals of the Quatro board, thus assigning each square either two or three colors (beads work well). A turn in Duo consists of one player designating an empty corner of the hcube, and the other player putting on it the beads from one Quatro square.

What is a win on the Duo board? A row of Quatro squares must all agree on some hcube feature. Each feature is represented by one of the eight cubes that compose the hcube. So, if four of the eight corners of any one cube all contain the same color, there is a win on the original Quatro board.

One easily can see when such a condition is threatened on the hcube, and also how a given move effects different dimensions. Other features of the hcube may also be used to assist your bookkeeping. When a face contains multiples of the same color, place that number of beads on its face proxy. This is useful, as a win requires at least two faces that each contain at least two of the same color. Conversely, if a face or cube can no longer be used in a win, a black bead on its proxy can remind you to ignore it.

4D in the Game of Set

Set is a deck of cards depicting all combinations of 4 feature-triples: red/green/purple, oval/diamond/squiggle, single/double/triple, and solid/hashed/hollow. 12 of these cards are dealt face up and players identify and remove "sets" as they become available. A set is any three cards that all match or all differ in each separate feature-domain. New cards are dealt to replace those removed in sets.

Although Set is a game of timing, the underlying structure can be analyzed. The Set cards can be mapped to points in a 3x3x3x3 grid. A set is any three points in a straight line of that grid. Diagonal represent an "all different" for that dimension. Note that these diagonals may wrap around, i.e., (1,1)(2,2)(3,3) works, but so does (2,1)(3,2)(1,3).

We may use our hcube diagram to represent this feature space by using the line, face, and cube proxies as well as the corners. This provides us with the needed 3-wide grid in each of the 4 dimensions. The hcube makes most 3-in-a-rows easy to spot. Grand diagonals through the center of the hcube can be difficult, but they also rarely occur.

4D Tic-Tac-Toe

Long the domain of the hypernerd, our new diagram makes this game actually playable. To provide the needed 4-wide grid, we use the corner points and the spaces *between* the center proxies. This works for most of the required nodes, but leaves a few (mostly interior) unaccounted for. A trick is needed for these, which we leave as an annoyance for the reader.

Thoughts

We used the trick of doubling a cube to create our hcube. The same can be done to create a fairly usable 5cube. Place two copies of our hcube side by side. These are the upper and lower layers of your new diagram. Each element in one is connected to its corresponding element in the other by an element one dimension higher, i.e., points are connected by edges, edges are connected by faces, faces are connected by cubes, and let's not forget, cubes are connected by hypercubes. For help, you might place a third hcube diagram between the other two to represent these elements, just as we sandwiched an extra element between every upper and lower pair in the hcube. Of course, elements on the middle hcube will have to stand for elements of one dimension higher, so perhaps you should get some crayons. And of course, the entire diagram had better encompass 10 hypercubes in all! We can see 8 of the cubes available to create hypercubes. Where are the remaining 2? We're sure you can find them…

The author is interested in hearing about other games that contain higher dimensions of information. I'd also like to know whether Quarto has been solved yet. At only 16-ply, the decision tree seems likely to be within reach.